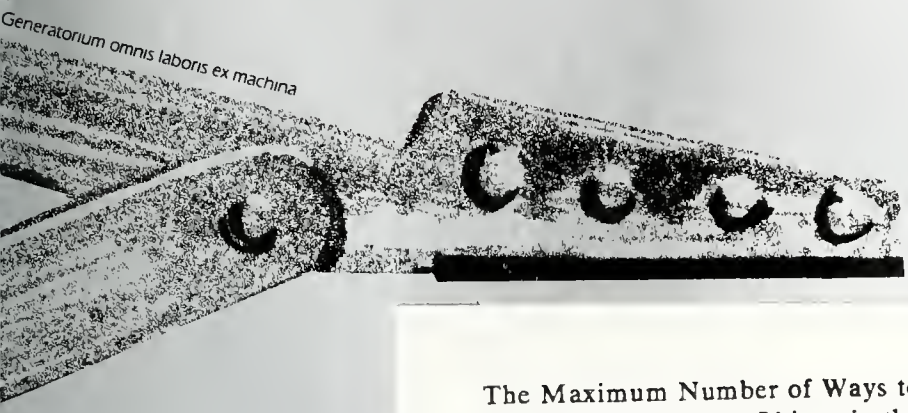
Robotics Research Technical Report

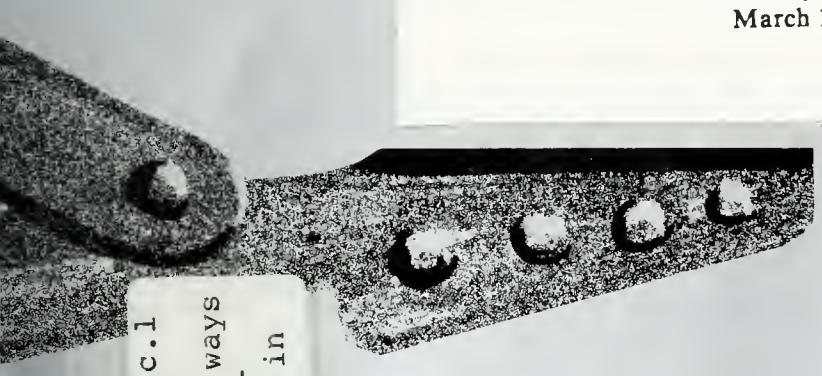


The Maximum Number of Ways to Stab n Convex Non-Intersecting Objects in the Plane 2n-2

by

Herbert Edelsbrunner
Micha Sharir

Technical Report No.281
Robotics Report No. 101
March 1987



NYU COMPSCI TR-281 c.1
Edelsbrunner, Herbert
The maximum number of ways
to stab n convex non-
intersecting objects in

Co nstitute of Mathematical Sciences

New York University

Computer Science Division
51 Mercer Street New York, N.Y. 10012

The Maximum Number of Ways to Stab n Convex Non-Intersecting Objects in the Plane 2n-2

by

Herbert Edelsbrunner
Micha Sharir

Technical Report No.281
Robotics Report No. 101
March 1987

New York University
Dept. of Computer Science
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, New York 10012

Research of the first author was supported by Amoco Fnd. Fac. Dev. Comput. Sci. 1-6-44862. Work by the second author has been supported by Office of Naval Research Grant N00014-82-K-0381, National Science Foundation CER Grant DCR-83-20085, and by grants from the Digital Equipment Corporation and the IBM Corporation.

THE MAXIMUM NUMBER OF WAYS TO STAB n CONVEX NON-INTERSECTING OBJECTS IN THE PLANE IS $2n-2$ [1]

Herbert Edelsbrunner[2] and Micha Sharir[3]

Abstract: Let S be a set of n convex, closed, and pairwise non-intersecting objects in the Euclidean plane labeled from 1 through n. A pair of permutations

$$((i_1,i_2,\ldots,i_{n-1},i_n),(i_n,i_{n-1},\ldots,i_2,i_1))$$

is called a *geometric permutation of* S if there is a line that intersects all objects of S in this order. We prove that S can realize at most $2n-2$ geometric permutations. This upper bound is tight.

Keywords: Combinatorial geometry, convexity, transversals, permutations, Davenport–Schinzel sequences.

---

[1] Research of the first author was supported by Amoco Fnd. Fac. Dev. Comput. Sci. 1-6-44862. Work on this paper by the second author was supported by Office of Naval Res. Grant N00014-82-K-0381, National Science Foundation Grant NSF-DCR-83-20085, and by grants from the Digital Equipment Corporation and the IBM Corporation.

[2]Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.

[3]Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, U.S.A., and School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel.

1. Introduction

Let S be a set of n convex, closed, bounded, and pairwise non-intersecting objects in the Euclidean plane. We label the objects with the integers from 1 through n. A directed or undirected line that intersects all objects is called a *transversal of* S. Since no two objects intersect each other, a transversal intersects S in a well-defined order. In the case of an undirected line, such an order can be described by a pair of permutations, one being the reverse of the other. Such a pair is called a *geometric permutation of* S. For convenience, we will represent a geometric permutation by any one of its two permutations.

For example, let S consist of two equally large disks, labeled $n-1$ and n, and of $n-2$ horizontal line segments, labeled from 1 through $n-2$. The two disks are placed sufficiently close to each other such that their centers lie on a common horizontal line h. The line segments are horizontal and lie above h such that their left endpoints lie sufficiently close to the boundary of the left disk and their right endpoints lie sufficiently close to the boundary of the right disk (see Figure 1). The labels of the line segments increase from the top downward. In this example, the geometric permutations are

$$(1,2,...,i,n-1,i+1,i+2,...,n-2,n), \text{ for } i=n-2,n-3,...,0, \text{ and}$$

$$(n-1,n-2,n-3,...,i+1,n,i,i-1,...,1), \text{ for } i=0,1,...,n-2.$$

Thus, we see that n objects can realize as many as $2n-2$ geometric permutations.

The above example is taken from [KLZ] which also proves that $\binom{n}{2}$ is an upper bound on the maximum number of geometric permutations realized by n convex non-intersecting objects. A proof of the same upper bound can also be found in [KLL]. An upper bound of $6n+6$ on the number of geometric permutations has

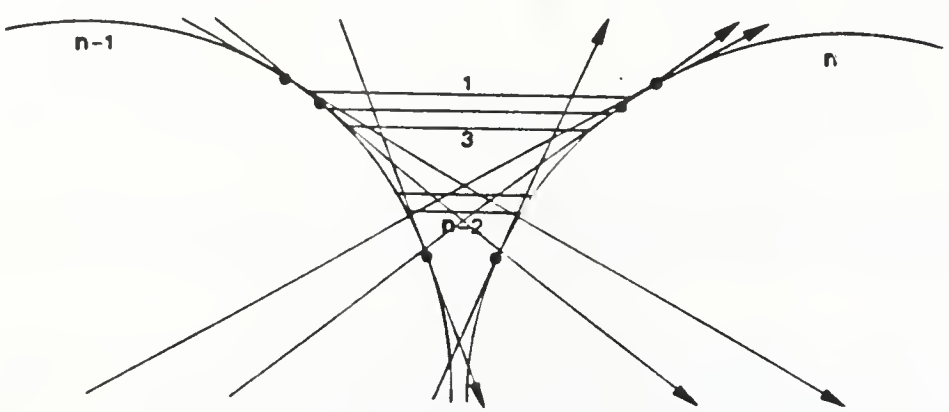


Figure 1. A set with $2n-2$ geometric permutations.

been shown recently in [W]. In this paper, we prove that the lower bound of $2n-2$ is in fact tight.

2. The Upper Bound

We derive the tight upper bound on the maximum number of geometric permutations by proving a sequence of three lemmas. First, we introduce a few definitions.

If a directed line $\vec{l}$ intersects all objects of S, then we say that $\vec{l}$ *induces* the permutation $(i_1,i_2,\ldots,i_n)$ that gives the order of objects along $\vec{l}$. Thus, the permutation induced by the line that coincides with $\vec{l}$ but has different direction is $(i_n,i_{n-1},\ldots,i_1)$. Similarly, for an undirected transversal, t, we say that t *induces* the pair of permutations that represent the order in which it intersects S. Line $\vec{l}$ is said to be *tangent to* object i if $\vec{l}$ contains a point on the boundary of i and i belongs to the closed half-plane to the left of $\vec{l}$. Note that this definition implies that two non-intersecting objects have no more than two common tangent directed lines.

For every angle $\alpha\in[0,2\pi)$, we define $\vec{l}(\alpha)$ as the unique directed line that satisfies the following three conditions.

(i) α is the angle between the positive x-axis and $\vec{l}(\alpha)$,

(ii) no object of S is contained in the open half-plane to the left of $\vec{l}(\alpha)$, and

(iii) at least one object of S is contained in the closed half-plane to the left of $\vec{l}(\alpha)$.

In Figures 1 and 2, lines $\vec{l}(\alpha)$ are shown for several angles α. It is clear from the definition of $\vec{l}(\alpha)$ that if there is a directed transversal of S with angle α, then $\vec{l}(\alpha)$ is the rightmost parallel transversal with this angle, that is, there is no transversal contained in the open half-plane to the right of $\vec{l}(\alpha)$. A line $\vec{l}(\alpha)$ of S is said to be *extreme* if it is tangent to at least two objects of S. For convenience, we also say that the undirected version of an extreme line is *extreme*. The significance of extreme lines derives from the following result.

Lemma 1: Every (undirected) transversal t of S can be moved continuously to an extreme line without ever changing the induced geometric permutation.

Proof: Let α_0 and $\beta_0=\alpha_0+\pi$ be the two angles of the two directed versions of t. Line t can be translated continuously to coincide with either $\vec{l}(\alpha_0)$ or $\vec{l}(\beta_0)$ without ever changing the induced geometric permutation.

Consider first the case that we move t to $\vec{l}(\alpha_0)$. Let i_0 be the (unique) object

contained in the closed half-plane to the left of $\vec{l}(\alpha_0)$ (see Figure 2). If this object is not unique, then $\vec{l}(\alpha_0)$ is an extreme line and we are done. Set $\alpha = \alpha_0$. Next, we decrease α continuously and move $\vec{l}(\alpha)$ accordingly until it ceases to be tangent to i_0 or it ceases to be a transversal of S. In the first case $\vec{l}(\alpha)$ passes through an extreme line in which case the lemma is proved. Otherwise, $\vec{l}(\alpha)$ passes a directed transversal $\vec{l}(\alpha_1)$ such that i_0 belongs to the closed half-plane to the left of $\vec{l}(\alpha_1)$ and there is another object, i_1, that belongs to the closed half-plane to the right of $\vec{l}(\alpha_1)$ (see Figure 2). We call this line an *inner tangent* of i_0 and i_1. If this is the case, we next increase α continuously (starting from α_0) until $\vec{l}(\alpha)$ ceases to be tangent to i_0 or it ceases to be a transversal of S. Again, this happens because $\vec{l}(\alpha)$ passes an extreme line or an inner tangent. Below, we consider the case that $\vec{l}(\alpha)$ does not reach the position of an extreme line; instead, it passes another inner tangent, $\vec{l}(\alpha_2)$, of i_0 and another object, i_2. Before we continue our argument note that we can assume that line $\vec{l}(\alpha_1)$ intersects the relative interior of every object except for that of i_0 and of i_1. Otherwise, $\vec{l}(\alpha_1)$ is an extreme line and we are done. The same assumption can be made for line $\vec{l}(\alpha_2)$.

Next, we translate t until it coincides with $\vec{l}(\beta_0)$. Let i_3 be the object contained in the closed half-plane to the left of $\vec{l}(\beta_0)$ which can be assumed to be unique. We may have i_3 equal to one of i_0, i_1, and i_2. Again, set $\beta = \beta_0$ and decrease β continuously until $\vec{l}(\beta)$ ceases to be tangent to i_3 or it ceases to be a transversal of S. We show below that the geometric constraints guarantee that $\vec{l}(\beta)$ reaches an extreme line during this process. These constraints are given by the fact that each line $\vec{l}(\alpha)$, $\alpha_1 \leq \alpha \leq \alpha_2$, meets all objects of S. Consequently, the intersection of the open half-planes to the right of $\vec{l}(\alpha_1)$ and $\vec{l}(\alpha_2)$ contains a point of each object in $S - \{i_0\}$ (see Figure 2). We denote this wedge by w. This is true by the following argument. If an object i_j, $j > 0$, intersects the closure of w but not w itself, then either $\vec{l}(\alpha_1)$ or $\vec{l}(\alpha_2)$ is an extreme line since it is tangent to i_j

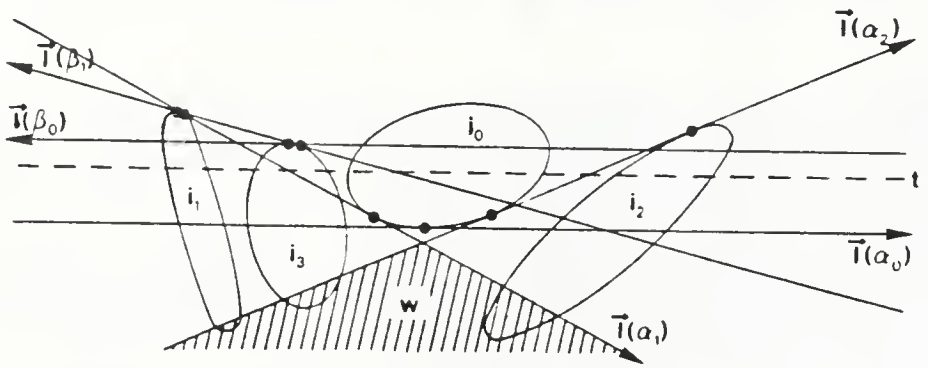


Figure 2. Moving a transversal to an extreme position.

and i_0. Thus, the only possibility left is that i_j avoids the closure of w. Still, i_j must intersect both $\vec{l}(\alpha_1)$ and $\vec{l}(\alpha_2)$. But this implies that there is an angle α, $\alpha_1 < \alpha < \alpha_2$, such that a line parallel to $\vec{l}(\alpha)$ separates i_j and i_0. This contradicts the fact that $\vec{l}(\alpha)$ is a transversal.

We can assume without loss of generality that $i_3 \neq i_1$. Since i_3 intersects $\vec{l}(\alpha_1)$, a line tangent to i_1, we know that $\vec{l}(\beta)$ intersects i_3 before i_1. Let $\vec{l}(\beta_1)$ be the common tangent of i_3 and i_1 that is reached first if we decrease the angle of $\vec{l}(\beta)$ and keep it tangent to i_3 (see Figure 2). Since $\vec{l}(\alpha_1)$ intersects the relative interior or i_3, line $\vec{l}(\beta_1)$ cannot intersect the open wedge w. By construction, also $\vec{l}(\beta_0)$ avoids w. Furthermore, at least one point of i_0 lies to the left of $\vec{l}(\beta_1)$, for example a point of i_0 that belongs to $\vec{l}(\alpha_1)$. This implies that every object has a point in the closed half-plane to the left of $\vec{l}(\beta_1)$, and, by construction, the same is true for line $\vec{l}(\beta_0)$. Consequently, no line $\vec{l}(\beta)$, $\beta_0 \leq \beta \leq \beta_1$ is an inner tangent of i_3 and another object. We deduce that $\vec{l}(\beta)$ passes an extreme line as we decrease β continuously - if no line $\vec{l}(\beta)$ is extreme for $\beta_0 \leq \beta < \beta_1$, then $\vec{l}(\beta_1)$ is extreme. □

Thus, an upper bound on the number of extreme lines puts also an upper bound on the number of geometric permutations. To derive such an upper bound, we associate S with a cyclic sequence of integers. Let $i(\alpha)$ be the object contained in the closed half-plane to the left of $\vec{l}(\alpha)$ - if it is unique. Since $\vec{l}(\alpha)$ is unique unless it is tangent to at least two objects, $i(\alpha)$ is defined except for at most a discrete number of angles α. We call the cyclic sequence

$$C(S) = i_1 i_2 \ldots i_m$$

($i_{m+1} = i_1$) the *cycle of* S if

(i) $i_j \neq i_{j+1}$, for $1 \leq j \leq m$, and

(ii) the circle of angles can be partitioned into m intervals $[\alpha_j, \alpha_{j+1})$, $1 \leq j \leq m$, such that $i(\alpha) = i_j$ if $\alpha \in (\alpha_j, \alpha_{j+1})$.

The *length of* the cycle of S is equal to m. The cycle of S can be constructed by rotating a directed line $\vec{l}$ through all lines $\vec{l}(\alpha)$ and by keeping track to which object $\vec{l}$ is tangent. A new object is reached whenever $\vec{l}$ passes an extreme line. Thus, m is also the number of extreme lines of S.

Below, we argue about scattered subcycles of $C(S)$ which are cyclic sequences that can be obtained from $C(S)$ by removing some of its members. The remaining integers appear in the same order as in $C(S)$.

Lemma 2: The cycle $C(S)$ contains no scattered subcycle of the form $abab$, with $a \neq b$.

Proof: Assume that there is a subcycle of the form $abab$. Then there are angles $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ such that

$$i(\alpha_1)=a,\ i(\alpha_2)=b,\ i(\alpha_3)=a,\ \text{and}\ i(\alpha_4)=b.$$

The definition of lines $\vec{l}(\alpha)$ and of labels $i(\alpha)$ implies that for each one of the open intervals (α_1,α_2), (α_2,α_3), (α_3,α_4), and (α_4,α_1), there is a common tangent of a and b whose angle is in this interval. However, since a and b do not intersect, there are only two directed lines that are tangent to a and b, a contradiction. □

Lemma 2 implies that if we remove all integers except for the a's and the b's from the cycle, then we get a cyclic sequence of the form

$$aa...abb...b.$$

We call a cyclic sequence that satisfies Lemma 2 and has no two equal integers in consecutive positions an $(n,2)$-*Davenport–Schinzel cycle* or an $(n,2)$-*cycle*, for short. Here, n refers to the largest number of different integers that occur in the cycle. By construction of $C(S)$, there is a one-to-one correspondence between the extreme lines of S and the pairs of consecutive integers in the cycle. Consequently, an upper bound on the maximum length of an $(n,2)$-cycle is also an upper bound on the maximum number of extreme lines.

Lemma 3: If $i_1 i_2 ... i_m$ is an $(n,2)$-cycle, then $m \leq 2n-2$.

Proof: The assertion is trivially true for $n=2$. Assume inductively that Lemma 3 holds for $(n-1,2)$-cycles. We prove below that every $(n,2)$-cycle has an integer a that occurs at most once. If this is true, then we can delete a and get an $(n-1,2)$-cycle after possibly removing also the predecessor of a – this has to be done if the predecessor and the successor of a are the same. Thus, every $(n,2)$-cycle contains at most two integers more than a longest $(n-1,2)$-cycle which implies the assertion.

To prove that there is always an integer a that occurs only once, we assume that such an integer does not exist. Let b be an integer that occurs at least twice, and let $i_j = b$ and $i_k = b$, $j<k$, be two successive appearances of b. By Lemma 2, every integer c that occurs between i_j and i_k (that is, $c = i_\ell$ with $j<\ell<k$) cannot occur outside this interval. For such an integer c, let $j(c)$ be the smallest index such that $i_{j(c)}=c$ and let z be the integer that maximizes this index. If z occurs again, then there must be another integer d between these two occurrences of z – and by definition of z, d also occurs between i_j and $i_{j(z)}$. But now, $dzdz$ is a subcycle of $C(S)$ which contradicts Lemma 2 and implies that there is always an integer a that makes only one appearance. □

We thus have shown that $2n-2$ is an upper bound on the number of extreme

lines. By Lemma 1, this implies that $2n-2$ is an upper bound on the number of geometric permutations of n convex, closed, bounded, and pairwise non-intersecting objects. If we reexamine the proofs of Lemmas 1 through 3, we realize that the only place where we use that the objects are bounded is where we assume that the lines $\vec{l}(\alpha)$ are well-defined. They are well-defined, however, even when S contains unbounded objects, provided there is no direction such that a transversal normal to that direction can be translated into this direction arbitrarily far without ceasing to be a transversal. For example, this is already the case when only one of the objects in S is bounded. If this condition is not satisfied, then $\vec{l}(\alpha)$ is not defined in a few non-degenerate intervals of angles α. All steps of the above development go through if we treat such a non-degenerate interval in the same way as we treat a single angle where $\vec{l}(\alpha)$ is not defined. Thus, we can dispense with the assumption that the objects are bounded altogether. This implies the main result of this paper.

Theorem: For $n \geq 2$, the maximum number of geometric permutations realized by n convex, closed, and pairwise non-intersecting objects in the plane is $2n-2$.

3. Discussion

Lemma 3, the upper bound on the length of $(n,2)$-cycles, can also be derived from the upper bound, $2n-1$, on the length of a so-called $(n,2)$-Davenport-Schinzel sequence, as proved in [A]. This is a sequence made up of at most n different integers that does not contain a scattered subsequence of the form *abab* and does not contain a pair of adjacent integers that are equal. It is interesting to note that $(n,2)$-cycles can be generalized to $(n,2s)$-cycles, $s \geq 1$, which contain no scattered subcycles of the form *abab...ab* of length $2s+2$. Upper and lower bounds on the maximum lengths of such cycles can be obtained by adapting the results in [HS, S1, S2].

The number of geometric permutations of a set S on n pairwise non-intersecting objects relates to the number of connected components of the interior of the so-called stabbing region as introduced in [EMPRWW]. The *stabbing region of* S is defined as the set of all points that are dual to transversals of S. Figure 3 shows a stabbing region of three line segments. Each line segment corresponds to a double wedge which is the area swept out by a rotating line – this line corresponds to a point that moves along the line segment. Centers of rotation are indicated by little dots in Figure 3. The duality transform used in the illustration relates vertical lines with points at infinity. Consequently, the double wedge corresponding to a bounded line segment does not contain a vertical line. We can see that the three double wedges in Figure 3 correspond to pairwise non-

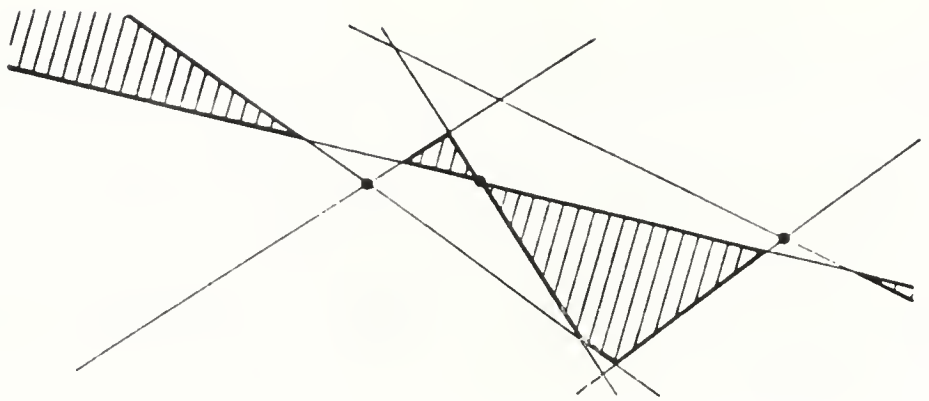


Figure 3. Stabbing region of three non-intersecting line segments.

intersecting line segments since no pair of double wedges shares a common line.

The upper bound on the number of extreme lines given in Lemmas 1 through 3 implies that the interior of the stabbing region of n convex and pairwise non-intersecting objects consists of at most $2n-1$ connected components – one more than the maximum number of extreme lines since it is possible that the two unbounded components, if they exist, contain only one point corresponding to an extreme line. In [EMPRWW], it is proved that the interior of the stabbing region of n line segments, intersecting or non-intersecting, consists of at most $n+1$ connected components. This implies that n non-intersecting line segments realize at most n geometric permutations – a result independently shown in [KLZ].

From the computational point of view it is interesting to ask how fast the geometric permutations of S can be computed. To answer this question, we assume that the objects in S are *computationally simple*, that is, we assume that

(i) the common tangents of two objects can be computed in constant time (this includes the two inner tangents of the objects),

(ii) given an angle α and an object i, we can compute the directed line with angle α that is tangent to i in constant time, and

(iii) given a directed line, we can determine which one of two objects it intersects first in constant time.

With these assumptions, we can use the divide-and-conquer approach that was exploited in [EMPRWW] to compute transversals for line segments and generalized in [AB] to arbitrary objects meeting some computational restrictions. Details about this approach can be found in those two papers. It leads to an algorithm that takes $O(n \log n)$ time to construct the stabbing region as defined above. Each connected component represents an equivalence class of lines that define the same geometric permutation – except for the two unbounded components which represent the same equivalence class, if they exist. To compute all geometric permutations of S, we recursively store the permutation with its associated

component. A component of the entire set is obtained as the intersection of two components computed recursively, and its permutation is obtained by merging the two permutations associated with the components. Using condition (iii), the merge step can be performed in linear time. Since the stabbing region consists of at most $2n-1$ connected components, this yields an $O(n^2)$ time algorithm. The algorithm is optimal in the worst case since it produces up to quadratic output.

Finally, there is the problem of generalizing the results to three and higher dimensions. It is shown in [W] that the number of geometric permutation defined by lines that intersect n convex objects in $d \geq 3$ dimensions is $O(n^{2d-2})$. A lower bound of $\Omega(n^{d-1})$ is claimed in [KLZ].

4. References

[A] Atallah, M. Dynamic computational geometry. *In* "Proc. 24th Ann. IEEE Sympos. Found. Comput. Sci. 1983", 92–99.

[AB] Atallah, M. and Bajaj, Ch. Efficient algorithms for common transversals. *Inform. Process. Lett.*, to appear.

[EMPRWW] Edelsbrunner, H., Maurer, H. A., Preparata, F. P., Rosenberg, A. L., Welzl, E. and Wood, D. Stabbing line segments. *BIT* **22** (1982), 274–281.

[HS] Hart, S. and Sharir, M. Nonlinearity of Davenport–Schinzel sequences and of a generalized path compression scheme. *Combinatorica* **6** (1986), 151–177.

[KLL] Katchalski, M., Lewis, T. and Liu, A. Geometric permutations and common transversals. *Discrete Comput. Geom.* **1** (1986), 371–377.

[KLZ] Katchalski, M., Lewis, T. and Zaks, J. Geometric permutations for convex sets. *Discrete Math.* **54** (1985), 271–284.

[S1] Sharir, M. Almost linear upper bounds on the length of general Davenport–Schinzel sequences. Rep. 29/1985, Eskenasy Inst. Comput. Sci., Tel Aviv Univ., Israel, 1985.

[S2] Sharir, M. Improved lower bounds on the length of Davenport–Schinzel sequences. Rep. 204, Courant Inst. Math. Sci., New York Univ., New York, 1986.

[W] Wenger, R. Upper bounds on linear orderings. Rep. TR-SOCS-86.19, School Comput. Sci., McGill Univ., Montreal, Quebec, 1986.

NYU COMPSCI TR-281 c.1
Edelsbrunner, Herbert
The maximum number of ways
to stab n convex non-
intersecting objects in

NYU COMPSCI TR-281 c.1
Edelsbrunner, Herbert
The maximum number of ways
to stab n convex non-
intersecting objects in

APR - 9 1987
DATE DUE

GAYLORD PRINTED IN U.S.A.